Week 6

6.1 Group Homomorphisms (cont'd)

Example 6.1.1.

- For any nonzero integer n, we have $n\mathbb{Z} < \mathbb{Z}$, and the map $\phi : n\mathbb{Z} \longrightarrow \mathbb{Z}$ defined by $nk \mapsto k$ is an isomorphism. Note that $n\mathbb{Z} < \mathbb{Z}$ is proper whenever $|n| > 1$, so a proper subgroup can be isomorphic to the parent group!

- On the other hand, for any integer n, the map $\phi : \mathbb{Z} \longrightarrow \mathbb{Z}$ defined by $k \mapsto nk$ is a homomorphism but *not* an isomorphism unless $|n| = 1$.

- Given a positive integer n, the remainder map $\phi : \mathbb{Z} \longrightarrow \mathbb{Z}_n$ defined by mapping k to its remainder when divided by n is a surjective homomorphism (check this!).

- The map $\phi : \mathbb{Z} \longrightarrow \mathbb{Z}$ defined by $k \mapsto k + 1$ is *not* a homomorphism.

Example 6.1.2. The group:

$$G = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \,\middle|\, \theta \in \mathbb{R} \right\}$$

is isomorphic to

$$G' = \{z \in \mathbb{C} : |z| = 1\}.$$

Here, the group operation on G is matrix multiplication, and the group operation on G' is the multiplication of complex numbers.

Proof. Each element in G' is equal to $e^{i\theta}$ for some $\theta \in \mathbb{R}$. Define a map $\phi : G \longrightarrow G'$ as follows:

$$\phi\left(\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \right) = e^{i\theta}.$$

Exercise: ϕ is a bijective group homomorphism. □

Here are some basic properties of group homomorphisms:

Proposition 6.1.3. *If $\phi : G \longrightarrow G'$ is a group homomorphism, then:*

1. $\phi(e_G) = e_{G'}$.
2. $\phi(g^{-1}) = \phi(g)^{-1}$, *for all* $g \in G$.
3. $\phi(g^n) = \phi(g)^n$, *for all* $g \in G$, $n \in \mathbb{Z}$.

Proof. We prove the first claim, and leave the rest as an exercise.

Since e_G is the identity element of G, we have $e_G * e_G = e_G$. On the other hand, since ϕ is a group homomorphism, we have:

$$\phi(e_G) = \phi(e_G * e_G) = \phi(e_G) *' \phi(e_G).$$

Since G' is a group, $\phi(e_G)^{-1}$ exists in G', hence:

$$\phi(e_G)^{-1} *' \phi(e_G) = \phi(e_G)^{-1} *' (\phi(e_G) *' \phi(e_G))$$

The left-hand side is equal to $e_{G'}$, while by the associativity of $*'$ the right-hand side is equal to $\phi(e_G)$. □

Let $\phi : G \longrightarrow G'$ be a homomorphism of groups. The **image** of ϕ is defined as:

$$\operatorname{im}\phi := \phi(G) := \{\phi(g) : g \in G\}$$

The **kernel** of ϕ is defined as:

$$\ker\phi = \{g \in G : \phi(g) = e_{G'}\}.$$

Proposition 6.1.4. *The image of ϕ is a subgroup of G'. The kernel of ϕ is a subgroup of G.*

Proof. **Exercise.** □

Proposition 6.1.5. *A group homomorphism $\phi : G \longrightarrow G'$ is one-to-one if and only if* $\ker\phi = \{e_G\}$.

Proof. **Exercise.** □

As we have mentioned, isomorphisms preserve algebraic properties. Here are some examples.

Proposition 6.1.6. *Let G be a cyclic group, then any group isomorphic to G is also cyclic.*

Proof. **Exercise.** $\square$

Example 6.1.7. The cyclic group $\mathbb{Z}_4$ is not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. Each element of $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ is of order at most 2. Since $|G| = 4$, G cannot be generated by any of its elements. Hence, G is not cyclic, so it cannot be isomorphic to the cyclic group $\mathbb{Z}_4$. $\square$

Proposition 6.1.8. *Let G be an abelian group, then any group isomorphic to G is abelian.*

Example 6.1.9. The group D_6 has 12 elements. We have seen that $D_6 = \langle r_2, s\rangle$, where r_2 is a rotation of order 6, and s is a reflection, which has order 2. So, it is reasonable to ask if D_6 is isomorphic to $\mathbb{Z}_6 \times \mathbb{Z}_2$. The answer is no. For $\mathbb{Z}_6 \times \mathbb{Z}_2$ is abelian, but D_6 is not.

Remark. Both claims remain true if we replace isomorphism by a surjective homomorphism, namely, if $\phi : G \longrightarrow G'$ is a surjective homomorphism, then we have

- G is cyclic $\Rightarrow G'$ is cyclic,
- G is abelian $\Rightarrow G'$ is abelian.

Try to prove these assertions by yourself!

Exercise. Check that the restriction of a homomorphism $\phi : G \longrightarrow G'$ to a subgroup $H < G$ gives a homomorphism from H to G'.

Proposition 6.1.10. *If $\phi : G \longrightarrow G'$ is an isomorphism, then $|\phi(g)| = |g|$ for any $g \in G$.*

Proof. By the previous exercise, the restriction of ϕ to the subgroup $\langle g\rangle$ gives a homomorphism

$$\phi|_{\langle g\rangle} : \langle g\rangle \longrightarrow G',$$

which is injective and with image

$$\operatorname{im}\phi|_{\langle g\rangle} = \langle \phi(g)\rangle.$$

So $\phi|_{\langle g\rangle}$ is an isomorphism from $\langle g\rangle$ to $\langle\phi(g)\rangle$; in particular, we have $|\phi(g)| = |g|$. $\square$